

**$E_n$ -REGULARITY IMPLIES  $E_{n-1}$ -REGULARITY**

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ABSTRACT. Vorst and latter Dayton-Weibel proved that  $K_n$ -regularity implies  $K_{n-1}$ -regularity. In this note we generalize this result from (commutative) rings to differential graded categories and from algebraic  $K$ -theory to any functor which is Morita invariant, continuous, and localizing. As an application, we show that the above implication also holds for schemes. Along the way, we extend Bass' fundamental theorem to this broader setting and prove a Nisnevich descent result which is of independent interest.

## 1. INTRODUCTION

Let  $n \in \mathbb{Z}$ . Following Bass [1, §XII], a (commutative) ring  $R$  is called  $K_n$ -regular if  $K_n(R) \simeq K_n(R[t_1, \dots, t_m])$  for all  $m \geq 1$ . Three decades ago, Vorst [22, Cor. 2.1] and latter Dayton-Weibel [7, Cor. 4.4] proved<sup>1</sup> the following implication

$$(1.1) \quad R \text{ is } K_n\text{-regular} \Rightarrow R \text{ is } K_{n-1}\text{-regular}.$$

It is then natural to ask the following:

*Question: Does implication (1.1) holds more generally ?*

**Statement of results.** A differential graded (=dg) category, over a base field  $k$ , is a category enriched over cochain complexes of  $k$ -vector spaces (morphisms sets are complexes) in such a way that composition fulfills the Leibniz rule  $d(f \circ g) = d(f) \circ g + (-1)^{\deg(f)} f \circ d(g)$ ; consult Keller's ICM survey [10]. Every (dg)  $k$ -algebra  $A$  gives naturally rise to a dg category  $\underline{A}$  with a single object and (dg)  $k$ -algebra of endomorphisms  $A$ . Another source of examples is provided by  $k$ -schemes. As explained by Lunts-Orlov [14] (see also [5, Example 4.5]), the derived category of perfect complexes of every (quasi-compact separated)  $k$ -scheme  $X$  admits a unique dg enhancement; which we will denote by  $\mathbf{perf}(X)$ .

Now, let  $E : \mathbf{dgc} \rightarrow \mathcal{T}$  be a functor, defined on the category of dg categories, and with values in an arbitrary triangulated category. The functor  $E$  is called:

- (i) *Morita invariant* if it inverts *Morita equivalences*, i.e. the dg functors  $\mathcal{A} \rightarrow \mathcal{B}$  inducing an equivalence  $\mathcal{D}(\mathcal{A}) \xrightarrow{\sim} \mathcal{D}(\mathcal{B})$  on derived categories; see [10, §3].
- (ii) *Continuous* if it preserves direct limits; see [16, Def. 1.6.4].
- (iii) *Localizing* if it sends the short exact sequences of dg categories (see [10, §4.6]) to distinguished triangles

$$0 \rightarrow \mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C} \rightarrow 0 \quad \mapsto \quad E(\mathcal{A}) \rightarrow E(\mathcal{B}) \rightarrow E(\mathcal{C}) \xrightarrow{\partial} \Sigma E(\mathcal{A}).$$

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<sup>1</sup>Vorst proved the cases  $n \geq 1$  and Dayton-Weibel the remaining cases  $n \leq 0$ .

Thanks to the work of Thomason-Trobaugh, Schlichting, Keller, Blumberg-Mandell and others (see [3, 12, 13, 17, 19, 20, 21]), examples of functors satisfying the above conditions (i)-(iii) include (nonconnective) algebraic  $K$ -theory ( $K$ ), Hochschild homology, cyclic homology (and its variants), topological Hochschild homology, *etc.* As proved in *loc. cit.*, when applied to  $\underline{A}$  (resp. to  $\mathbf{perf}(X)$ ) these functors reduce to the classical invariants of (dg)  $k$ -algebras (resp. of  $k$ -schemes).

Given a dg category  $\mathcal{A}$ , an integer  $n$ , a functor  $E : \mathbf{dgc} \rightarrow \mathcal{T}$ , and an object  $b \in \mathcal{T}$ , let us write  $E_n^b(\mathcal{A})$  for the abelian group  $\mathrm{Hom}_{\mathcal{T}}(\Sigma^n b, E(\mathcal{A}))$ . For instance, when  $\mathcal{A} = \underline{A}$ ,  $E = K$  and  $b = \mathbb{S}$  is the sphere spectrum,  $E_n^b(\mathcal{A})$  identifies with the  $n^{\mathrm{th}}$  algebraic  $K$ -theory group of  $A$ . Following Bass, let us then call a dg category  $\mathcal{A}$   $E_n^b$ -regular if  $E_n^b(\mathcal{A}) \simeq E_n^b(\mathcal{A}[t_1, \dots, t_m])$  for all  $m \geq 1$ , where  $\mathcal{A}[t_1, \dots, t_m] := \mathcal{A} \otimes k[t_1, \dots, t_m]$ . Our main result, which answers affirmatively to the above question, is the following:

**Theorem 1.2.** *Let  $\mathcal{A}$  be a dg category,  $n$  and integer,  $E : \mathbf{dgc} \rightarrow \mathcal{T}$  a functor satisfying the above conditions (i)-(iii), and  $b$  a compact object of  $\mathcal{T}$ . Under these notations and assumptions, the following implication hold:*

$$\mathcal{A} \text{ is } E_n^b\text{-regular} \Rightarrow \mathcal{A} \text{ is } E_{n-1}^b\text{-regular}.$$

Note that Theorem 1.2 uncovers in a direct an elegant way the three key conceptual properties (= Morita invariance, continuity, and localization) that underlie Vorst and Dayton-Weibel's implication (1.1). Along its proof, we have generalized Bass' fundamental theorem and introduced a Nisnevich descent result; see Theorems 3.1 and 4.2. These results are of independent interest.

Following Bass, a (quasi-compact separated)  $k$ -scheme  $X$  is called  $K_n$ -regular if  $K_n(X) \simeq K_n(X \times \mathbb{A}^m)$  for all  $m \geq 1$ , where  $\mathbb{A}$  stands for the affine line. As mentioned above, all the invariants of  $X$  can be recovered from its dg derived category of perfect complexes  $\mathbf{perf}(X)$ . Hence, let us define  $E_n^b(X)$  to be the abelian group  $E_n^b(\mathbf{perf}(X))$  and let us call a  $k$ -scheme  $X$   $E_n^b$ -regular if  $E_n^b(X) \simeq E_n^b(X \times \mathbb{A}^m)$  for all  $m \geq 1$ . As an application of Theorem 1.2 we obtain the following:

**Theorem 1.3.** *Let  $X$  be a quasi-compact separated  $k$ -scheme,  $n$  and integer,  $E : \mathbf{dgc} \rightarrow \mathcal{T}$  a functor satisfying the above conditions (i)-(iii), and  $b$  a compact object of  $\mathcal{T}$ . Under these notations and assumptions, the following implication hold:*

$$(1.4) \quad X \text{ is } E_n^b\text{-regular} \Rightarrow X \text{ is } E_{n-1}^b\text{-regular}.$$

When  $E = K$  and  $b = \mathbb{S}$ , (1.4) reduces to  $K_n$ -regularity  $\Rightarrow K_{n-1}$ -regularity. Chuck Weibel kindly informed the author that this latter implication was proved (in a totally different way) by Cortiñas-Haesemeyer-Walker-Weibel [6, Cor. 4.4] in the particular case where  $k$  is of characteristic zero. To the best of the author's knowledge the remaining cases in positive characteristic are new in the literature.

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## 2. NOTATIONS

Throughout this note we will reserve the letter  $k$  for our base field. All  $k$ -schemes will be assumed to be quasi-compact and separated. Given a dg category  $\mathcal{A}$  and a  $k$ -scheme  $X$ , we will often write  $\mathcal{A} \otimes X$  instead of  $\mathcal{A} \otimes \mathbf{perf}(X)$ . When  $X = \mathrm{spec}(A)$  is affine we will furthermore replace  $\mathcal{A} \otimes \mathrm{spec}(A)$  by  $\mathcal{A} \otimes \underline{A}$ . Finally, the category of

dg categories with the Morita equivalences inverted will be denoted by  $\mathrm{Ho}(\mathrm{dgc}at)$ . Note that every Morita invariant functor descends uniquely to  $\mathrm{Ho}(\mathrm{dgc}at)$ .

### 3. NISNEVICH DESCENT

In this section we prove the following Nisnevich descent result, which is of independent interest. Its Corollary 3.6 will play a key role in the next section.

**Theorem 3.1.** (*Nisnevich descent*) *Consider the following (distinguished) square of smooth  $k$ -schemes*

$$(3.2) \quad \begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{j} & X, \end{array}$$

where  $j$  is an open immersion and  $p$  is an étale morphism inducing an isomorphism of reduced  $k$ -schemes  $p^{-1}(X - U)_{\mathrm{red}} \simeq (X - U)_{\mathrm{red}}$ . Then, for every dg category  $\mathcal{A}$  and Morita invariant localizing functor  $E : \mathrm{dgc}at \rightarrow \mathcal{T}$ , one obtains a homotopy (co)cartesian square

$$(3.3) \quad \begin{array}{ccc} E(\mathcal{A} \otimes X) & \xrightarrow{E(\mathrm{id} \otimes j^*)} & E(\mathcal{A} \otimes U) \\ E(\mathrm{id} \otimes p^*) \downarrow & \square & \downarrow \\ E(\mathcal{A} \otimes V) & \longrightarrow & E(\mathcal{A} \otimes (U \times_X V)) \end{array}$$

in the triangulated category  $\mathcal{T}$ ; see [16, Def. 1.4.1].

*Proof.* Consider the following commutative diagram in  $\mathrm{Ho}(\mathrm{dgc}at)$

$$(3.4) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{perf}(X)_Z & \longrightarrow & \mathrm{perf}(X) & \xrightarrow{j^*} & \mathrm{perf}(U) \longrightarrow 0 \\ & & \simeq \downarrow & & p^* \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{perf}(V)_{Z'} & \longrightarrow & \mathrm{perf}(V) & \longrightarrow & \mathrm{perf}(U \times_X V) \longrightarrow 0, \end{array}$$

where  $Z$  (resp.  $Z'$ ) is the closed set  $X - U$  (resp.  $p^{-1}(X - U)$ ) and  $\mathrm{perf}(X)_Z$  (resp.  $\mathrm{perf}(V)_{Z'}$ ) the dg category of those perfect complexes of  $\mathcal{O}_X$ -modules (resp. of  $\mathcal{O}_V$ -modules) that are supported on  $Z$  (resp. on  $Z'$ ). As explained by Thomason-Trobaugh in [21, §5], both rows are short exact sequences of dg categories; see also [10, §4.6]. Furthermore, as proved in [21, Thm. 2.6.3], the induced dg functor  $\mathrm{perf}(X)_Z \rightarrow \mathrm{perf}(V)_{Z'}$  is a Morita equivalence.

Now, since we are working over a field  $k$ , every dg category  $\mathcal{A}$  is  $k$ -flat, i.e. for every pair of objects  $(x, y)$  the functor  $\mathcal{A}(x, y) \otimes -$  preserves quasi-isomorphisms; see [10, §4.2]. As a consequence, the functor  $\mathcal{A} \otimes -$  preserves Morita equivalences and, as proved by Drinfeld in [8, Prop. 1.6.3], short exact sequences of dg categories. Hence, by first tensoring (3.4) with  $\mathcal{A}$  and then applying the functor  $E$  we obtain the following commutative diagram:

$$\begin{array}{ccccccc} E(\mathcal{A} \otimes \mathrm{perf}(X)_Z) & \longrightarrow & E(\mathcal{A} \otimes X) & \xrightarrow{E(\mathrm{id} \otimes j^*)} & E(\mathcal{A} \otimes U) & \xrightarrow{\partial} & \Sigma E(\mathcal{A} \otimes \mathrm{perf}(X)_Z) \\ \simeq \downarrow & & E(\mathrm{id} \otimes p^*) \downarrow & & \downarrow & & \downarrow \simeq \\ E(\mathcal{A} \otimes \mathrm{perf}(V)_{Z'}) & \longrightarrow & E(\mathcal{A} \otimes V) & \longrightarrow & E(\mathcal{A} \otimes (U \times_X V)) & \xrightarrow{\partial} & \Sigma E(\mathcal{A} \otimes \mathrm{perf}(V)_{Z'}). \end{array}$$

Since each row is a distinguished triangle and the outer left and right vertical maps are isomorphisms we conclude that the middle square (which agrees with (3.3)) is homotopy (co)cartesian. This achieves the proof.  $\square$

*Remark 3.5.* Let us denote by  $\mathrm{Sm}(k)$  the category of smooth  $k$ -schemes. By combining Theorem 3.1 with [15, §3.1, Prop. 1.4] one observes that the following presheaf<sup>2</sup>

$$\mathrm{Sm}(k)^{\mathrm{op}} \xrightarrow{\mathrm{perf}(-)} \mathrm{Ho}(\mathrm{dgc}at) \xrightarrow{\mathcal{A} \otimes -} \mathrm{Ho}(\mathrm{dgc}at) \xrightarrow{E} \mathcal{T}$$

is in fact a sheaf with respect to the Nisnevich topology on  $\mathrm{Sm}(k)$ .

**Corollary 3.6.** (*Mayer-Vietoris for open covers*) *Let  $X$  be a smooth  $k$ -scheme which is covered by two Zariski open subschemes  $U, V \subset X$ . Then, for every dg category  $\mathcal{A}$  and Morita invariant localizing functor  $E : \mathrm{dgc}at \rightarrow \mathcal{T}$ , one obtains a Mayer-Vietoris triangle*

$$E(\mathcal{A} \otimes X) \rightarrow E(\mathcal{A} \otimes U) \oplus E(\mathcal{A} \otimes V) \xrightarrow{\pm} E(\mathcal{A} \otimes (U \cap V)) \xrightarrow{\partial} \Sigma E(\mathcal{A} \otimes X).$$

*Proof.* This follows from the fact that when the morphism  $p$  in (3.2) is a open immersion,  $U \times_X V$  identifies with  $U \cap V$ ; recall also from [16, §1.4] that every homotopy (co)cartesian square has an associated distinguished triangle.  $\square$

#### 4. GENERALIZED FUNDAMENTAL THEOREM

Bass proved in [1, §XII-§7] the following fundamental result:

**Theorem 4.1.** (*Bass' fundamental theorem*) *Let  $R$  be a ring and  $n$  an integer. Then, one has the following exact sequence of abelian groups*

$$0 \rightarrow K_n(R) \xrightarrow{\Delta} K_n(R[x]) \oplus K_n(R[1/x]) \xrightarrow{\pm} K_n(R[x, 1/x]) \xrightarrow{\partial_n} K_{n-1}(R) \rightarrow 0.$$

In this section we generalize it as follows:

**Theorem 4.2.** (*Generalized fundamental theorem*) *Let  $\mathcal{A}$  be a dg category,  $n$  an integer,  $E : \mathrm{dgc}at \rightarrow \mathcal{T}$  a Morita invariant localizing functor, and  $b$  an object of  $\mathcal{T}$ . Then, one has the following exact sequence of abelian groups*

$$(4.3) \quad 0 \rightarrow E_n^b(\mathcal{A}) \xrightarrow{\Delta} E_n^b(\mathcal{A}[x]) \oplus E_n^b(\mathcal{A}[1/x]) \xrightarrow{\pm} E_n^b(\mathcal{A}[x, 1/x]) \xrightarrow{\partial_n} E_{n-1}^b(\mathcal{A}) \rightarrow 0.$$

*Remark 4.4.* Consult Remark 6.3 for the instantiation of (4.3) to  $k$ -schemes.

*Proof.* Let  $\mathbb{P}_k^1$  be the projective line and  $i : \mathrm{spec}(k[x]) \subset \mathbb{P}_k^1$  and  $j : \mathrm{spec}(k[1/x]) \subset \mathbb{P}_k^1$  its classical Zariski open cover. Since  $\mathrm{spec}(k[x]) \cap \mathrm{spec}(k[1/x]) = \mathrm{spec}(k[x, 1/x])$ , one obtains from the above Corollary 3.6 the following distinguished triangle

$$E(\mathcal{A} \otimes \mathbb{P}_k^1) \xrightarrow{(E(\mathrm{id} \otimes i^*), E(\mathrm{id} \otimes j^*))} E(\mathcal{A}[x]) \oplus E(\mathcal{A}[1/x]) \xrightarrow{\pm} E(\mathcal{A}[x, 1/x]) \xrightarrow{\partial} \Sigma E(\mathcal{A} \otimes \mathbb{P}_k^1).$$

Let us now study the object  $E(\mathcal{A} \otimes \mathbb{P}_k^1)$ . As proved by Beilinson [2], the dg category  $\mathrm{perf}(\mathbb{P}_k^1)$  admits a full exceptional collection (see [9, Def. 1.57]) given by the line bundles  $\mathcal{O}_{\mathbb{P}_k^1}(0)$  and  $\mathcal{O}_{\mathbb{P}_k^1}(-1)$ . Consequently, the fully faithful dg functors

$$\iota_0 : \mathrm{perf}(k) \rightarrow \mathrm{perf}(\mathbb{P}_k^1) \quad k \mapsto \mathcal{O}_{\mathbb{P}_k^1}(0) \quad \iota_{-1} : \mathrm{perf}(k) \rightarrow \mathrm{perf}(\mathbb{P}_k^1) \quad k \mapsto \mathcal{O}_{\mathbb{P}_k^1}(-1)$$

<sup>2</sup>i.e. a contravariant functor from  $\mathrm{Sm}(k)$  to  $\mathcal{T}$ .

give rise to a semi-orthogonal decomposition  $\text{perf}(\mathbb{P}_k^1) = \langle \text{perf}(k), \text{perf}(k) \rangle$ ; see [9, Def. 1.59]. As explained in [18, §3], we obtain then a well-defined *split* short exact sequence of dg categories

$$(4.5) \quad 0 \longrightarrow \text{perf}(k) \xrightarrow[\iota_0]{r} \text{perf}(\mathbb{P}_k^1) \xrightarrow[s]{\iota_{-1}} \text{perf}(k) \longrightarrow 0,$$

where  $r$  is the right adjoint of  $\iota_0$ ,  $r \circ \iota_0 = \text{id}$ ,  $\iota_{-1}$  is right adjoint of  $s$ , and  $\iota_{-1} \circ s = \text{id}$ . Note that split short exact sequences are stable under tensorization with a dg category and that every localizing functor sends split short exact sequences to split distinguished triangles, i.e. to direct sums in  $\mathcal{T}$ . Hence, by first tensoring (4.5) with  $\mathcal{A}$  and then applying the functor  $E$  we obtain the isomorphism

$$(4.6) \quad (E(\text{id} \otimes \iota_0), E(\text{id} \otimes \iota_{-1})) : E(\mathcal{A} \otimes \underline{k}) \oplus E(\mathcal{A} \otimes \underline{k}) \xrightarrow{\sim} E(\mathcal{A} \otimes \mathbb{P}_k^1).$$

Recall that the line bundles  $\mathcal{O}_{\mathbb{P}_k^1}(0)$  and  $\mathcal{O}_{\mathbb{P}_k^1}(-1)$  become isomorphic when restricted to  $\text{spec}(k[x])$  and  $\text{spec}(k[1/x])$ . Hence, we have the commutative diagrams

$$\begin{array}{ccc} \text{perf}(k) & \xrightleftharpoons[\iota_{-1}]{\iota_0} & \text{perf}(\mathbb{P}_k^1) \xrightarrow{i^*} \text{perf}(\text{spec}(k[x])) \\ \text{perf}(k) & \xrightleftharpoons[\iota_{-1}]{\iota_0} & \text{perf}(\mathbb{P}_k^1) \xrightarrow{j^*} \text{perf}(\text{spec}(k[1/x])) \end{array}$$

and consequently we obtain the equalities:

$$(4.7) \quad E(\text{id} \otimes i^*) \circ E(\text{id} \otimes \iota_0) = E(\text{id} \otimes i^*) \circ E(\text{id} \otimes \iota_{-1})$$

$$(4.8) \quad E(\text{id} \otimes j^*) \circ E(\text{id} \otimes \iota_0) = E(\text{id} \otimes j^*) \circ E(\text{id} \otimes \iota_{-1}).$$

Now, apply Lemma 4.9 to isomorphism (4.6) and then compose the result with  $(E(\text{id} \otimes i^*), E(\text{id} \otimes j^*))$ . Thanks to (4.7)-(4.8), we obtain a morphism

$$\Psi : E(\mathcal{A} \otimes \underline{k}) \oplus E(\mathcal{A} \otimes \underline{k}) \longrightarrow E(\mathcal{A}[x]) \oplus E(\mathcal{A}[1/x]),$$

which is  $E(\text{id} \otimes i^*) \circ E(\text{id} \otimes \iota_0)$  on the first component and zero on the second component. Making use of it, the above distinguished triangle identifies with

$$E(\mathcal{A}) \oplus E(\mathcal{A}) \xrightarrow{\Psi} E(\mathcal{A}[x]) \oplus E(\mathcal{A}[1/x]) \xrightarrow{\pm} E(\mathcal{A}[x, 1/x]) \xrightarrow{\partial} \Sigma E(\mathcal{A}) \oplus \Sigma E(\mathcal{A}).$$

By applying to it the functor  $\text{Hom}_{\mathcal{T}}(\Sigma^n b, -)$  we obtain then a long exact sequence

$$\begin{aligned} \cdots \rightarrow E_n^b(\mathcal{A}) \oplus E_n^b(\mathcal{A}) &\xrightarrow{\Psi_n} E_n^b(\mathcal{A}[x]) \oplus E_n^b(\mathcal{A}[1/x]) \xrightarrow{\pm} E_n^b(\mathcal{A}[x, 1/x]) \longrightarrow \\ &\xrightarrow{\partial_n} E_{n-1}^b(\mathcal{A}) \oplus E_{n-1}^b(\mathcal{A}) \xrightarrow{\Psi_{n-1}} E_{n-1}^b(\mathcal{A}[x]) \oplus E_{n-1}^b(\mathcal{A}[1/x]) \xrightarrow{\pm} E_{n-1}^b(\mathcal{A}[x, 1/x]) \rightarrow \cdots \end{aligned}$$

As explained above,  $\Psi_n$  is zero when restricted to the second component. Moreover, the retraction  $k[x] \rightarrow k, x \mapsto 0$ , to the inclusion  $k \subset k[x]$  shows us that  $\Psi_n$  is injective when restricted to the first component. This implies that the image of  $\partial_n$  is precisely the second component of the direct sum. As a consequence, the above long exact sequence breaks up into the exact sequences (4.3). This achieves the proof.  $\square$

**Lemma 4.9.** *If  $(f, g) : A \oplus A \xrightarrow{\sim} B$  is an isomorphism in an additive category, then  $(f, f - g) : A \oplus A \xrightarrow{\sim} B$  is also an isomorphism.*

*Proof.* Since  $(f, g)$  is an isomorphism, there exist maps  $i, h : B \rightarrow A$  such that  $fi + gh = \text{id}$ ,  $if = \text{id}$ ,  $hf = 0$ ,  $ig = 0$ , and  $hg = \text{id}$ . Using these equalities one observes that  $(i + h, -h) : B \xrightarrow{\sim} A \oplus A$  is the inverse of  $(f, f - g)$ .  $\square$

**Notation 4.10.** Given a dg category  $\mathcal{A}$ , let us denote by  $NE_n^b(\mathcal{A})$  the kernel of the surjective group homomorphism

$$(4.11) \quad E_n^b(\text{id} \otimes (t = 0)) : E_n^b(\mathcal{A}[t]) \longrightarrow E_n^b(\mathcal{A}).$$

Note that the inclusion  $k \subset k[t]$  gives rise to a direct sum decomposition  $E_n^b(\mathcal{A}[t]) \simeq NE_n^b(\mathcal{A}) \oplus E_n^b(\mathcal{A})$ . Note also that by induction on  $m$ ,  $\mathcal{A}$  is  $E_n^b$ -regular if and only if  $NE_n^b(\mathcal{A}[t_1, \dots, t_{m-1}][t_m]) = 0$  for all  $m \geq 0$ .

**Corollary 4.12.** *Under the notations and assumptions of Theorem 4.2, we have the following exact sequence of abelian groups*

$$0 \rightarrow NE_n^b(\mathcal{A}) \xrightarrow{\Delta} NE_n^b(\mathcal{A}[x]) \oplus NE_n^b(\mathcal{A}[1/x]) \xrightarrow{\pm} NE_n^b(\mathcal{A}[x, 1/x]) \xrightarrow{\partial_n} NE_{n-1}^b(\mathcal{A}) \rightarrow 0.$$

*Proof.* This follows automatically from the naturality of (4.3).  $\square$

## 5. PROOF OF THEOREM 1.2

Consider the following “substitution”  $k$ -algebra homomorphism

$$(5.1) \quad k[x][t] \longrightarrow k[x][t] \quad p(x, t) \mapsto p(x, xt).$$

Given a dg category  $\mathcal{B}$ , let us denote by  $\text{colim } NE_n^b(\mathcal{B}[x])$  the direct limit of the following diagram of abelian groups

$$NE_n^b(\mathcal{B}[x]) \xrightarrow{NE_n^b(\text{id} \otimes (5.1))} NE_n^b(\mathcal{B}[x]) \xrightarrow{NE_n^b(\text{id} \otimes (5.1))} NE_n^b(\mathcal{B}[x]) \xrightarrow{NE_n^b(\text{id} \otimes (5.1))} \dots$$

We start by proving that we one has a group isomorphism

$$(5.2) \quad \text{colim } NE_n^b(\mathcal{B}[x]) \simeq NE_n^b(\mathcal{B}[x, x^{-1}]).$$

Consider first the commutative diagram

$$(5.3) \quad \begin{array}{ccccccc} k[x][t] & \xrightarrow{(5.1)} & k[x][t] & \xrightarrow{(5.1)} & k[x][t] & \xrightarrow{(5.1)} & \dots \\ (t=0) \downarrow & & (t=0) \downarrow & & (t=0) \downarrow & & \\ k[x] & \xlongequal{\quad} & k[x] & \xlongequal{\quad} & k[x] & \xlongequal{\quad} & \dots \end{array}$$

Note that the colimit of the lower row is  $k[x]$  while the colimit of the upper row is the  $k$ -algebra  $R := k[x] + tk[x, 1/x][t] \subset k[x, 1/x][t]$ . By first tensoring (5.3) with  $\mathcal{B}$  and then applying the functor  $E_n^b$  we obtain the commutative diagram

$$(5.4) \quad \begin{array}{ccccccc} NE_n^b(\mathcal{B}[x]) & \xrightarrow{NE_n^b(\text{id} \otimes (5.1))} & NE_n^b(\mathcal{B}[x]) & \xrightarrow{NE_n^b(\text{id} \otimes (5.1))} & NE_n^b(\mathcal{B}[x]) & \xrightarrow{NE_n^b(\text{id} \otimes (5.1))} & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ E_n^b(\mathcal{B}[x][t]) & \xrightarrow{E_n^b(\text{id} \otimes (5.1))} & E_n^b(\mathcal{B}[x][t]) & \xrightarrow{E_n^b(\text{id} \otimes (5.1))} & E_n^b(\mathcal{B}[x][t]) & \xrightarrow{E_n^b(\text{id} \otimes (5.1))} & \dots \\ (4.11) \downarrow & & (4.11) \downarrow & & (4.11) \downarrow & & \\ E_n^b(\mathcal{B}[x]) & \xlongequal{\quad} & E_n^b(\mathcal{B}[x]) & \xlongequal{\quad} & E_n^b(\mathcal{B}[x]) & \xlongequal{\quad} & \dots \end{array}$$

Recall from Notation (4.10) that each column is a (split) short exact sequence of abelian groups. The colimit of the lower row is clearly  $E_n^b(\mathcal{B}[x])$ . Since the functors  $\mathcal{B} \otimes - : \text{dgc} \rightarrow \text{dgc}$  and  $E : \text{dgc} \rightarrow \mathcal{T}$  preserve direct limits and  $b$  is a compact

object of  $\mathcal{T}$ , the colimit of the middle row identifies with  $E_n^b(\mathcal{B} \otimes \underline{R})$ . Hence, from diagram (5.4) one obtains the isomorphism

$$(5.5) \quad \operatorname{colim} NE_n^b(\mathcal{B}[x]) \simeq \operatorname{Ker}(E_n^b(\mathcal{B} \otimes \underline{R}) \xrightarrow{(4.11)} E_n^b(\mathcal{B}[x])) .$$

Now, consider the  $k$ -algebras  $R$  and  $k[x]$  endowed with the sets of left denominators  $S_1 := \{x^n\}_{n \geq 0} \subset R$  and  $S_2 := \{x^n\}_{n \geq 0} \subset k[x]$ . The  $k$ -algebra homomorphism

$$(5.6) \quad R = k[x] + tk[x, 1/x][t] \longrightarrow k[x] \quad t \mapsto 0$$

identifies  $S_1$  with  $S_2$  and moreover induces a quasi-isomorphism

$$\begin{array}{ccccccc} 0 & \longrightarrow & R & \longrightarrow & R[S_1^{-1}] = k[x, 1/x][t] & \longrightarrow & 0 \\ & & \downarrow (5.6) & & \downarrow (5.6) & & \\ 0 & \longrightarrow & k[x] & \longrightarrow & k[t][S_2^{-1}] = k[x, 1/x] & \longrightarrow & 0 . \end{array}$$

As a consequence, conditions a) and b) of [11, §4.2] are satisfied and so one obtains a well-defined commutative diagram

$$(5.7) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \underline{A_1} & \longrightarrow & \operatorname{perf}(R) & \longrightarrow & \operatorname{perf}(k[x, x^{-1}][t]) \longrightarrow 0 \\ & & \simeq \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \underline{A_2} & \longrightarrow & \operatorname{perf}(k[x]) & \longrightarrow & \operatorname{perf}(k[x, x^{-1}]) \longrightarrow 0 , \end{array}$$

where each row is a short exact sequence of dg categories and the left vertical map is a quasi-isomorphism (and hence a Morita equivalence) of dg  $k$ -algebras; consult [11, §4.3] for further details. By first tensoring (5.7) with  $\mathcal{B}$  and then applying the functor  $E$  we obtain (as in the proof of Theorem 3.1) a homotopy (co)cartesian square

$$(5.8) \quad \begin{array}{ccc} E(\mathcal{B} \otimes \underline{R}) & \longrightarrow & E(\mathcal{B}[x, 1/x][t]) \\ \downarrow & \square & \downarrow \\ E(\mathcal{B}[x]) & \longrightarrow & E(\mathcal{B}[x, 1/x]) . \end{array}$$

Note that the natural inclusions  $k[x] \subset R$  and  $k[x, 1/x] \subset k[x, 1/x][t]$  give rise to sections of the vertical maps. As a consequence, since (5.8) is homotopy (co)cartesian, we obtain an induced isomorphism

$$\operatorname{Ker}(E_n^b(\mathcal{B} \otimes \underline{R}) \xrightarrow{(4.11)} E_n^b(\mathcal{B}[x])) \xrightarrow{\simeq} \operatorname{Ker}(E_n^b(\mathcal{B}[x, 1/x][t]) \xrightarrow{(4.11)} E_n^b(\mathcal{B}[x, 1/x])) .$$

Since the right-hand-side is by definition  $NE_n^b(\mathcal{B}[x, 1/x])$  the searched isomorphism (5.2) follows now from isomorphism (5.5).

We are now ready to conclude the proof. As explained in Notation 4.10, a dg category  $\mathcal{A}$  is  $E_n^b$ -regular if and only if  $NE_n^b(\mathcal{A}[t_1, \dots, t_{m-1}][t_m]) = 0$  for any all  $m \geq 0$ . Since by hypothesis  $\mathcal{A}$  is  $E_n^b$ -regular we then have  $NE_n^b(\mathcal{A}[t_1, \dots, t_{m-1}][t_m]) = 0$  for all  $m \geq 0$ . Using isomorphism (5.2) (with  $\mathcal{B} = \mathcal{A}[t_1, \dots, t_{m-1}]$ ) we conclude that

$$\operatorname{colim} NE_n^b(\mathcal{A}[t_1, \dots, t_{m-1}][x]) \simeq NE_n^b(\mathcal{A}[t_1, \dots, t_{m-1}][x, 1/x]) = 0 .$$

The exact sequence of Corollary 4.12 (with  $\mathcal{A} = \mathcal{A}[t_1, \dots, t_{m-1}]$ ) allows us to conclude that  $NE_{n-1}^b(\mathcal{A}[t_1, \dots, t_{m-1}]) = 0$ . Since this holds for every  $m \geq 0$ , we then conclude that  $\mathcal{A}$  is  $E_{n-1}^b$ -regular. The proof of Theorem 1.2 is then finished.

## 6. PROOF OF THEOREM 1.3

Since  $X$  is  $E_n^b$ -regular the isomorphism  $E_n^b(X) \simeq E_n^b(X \times \mathbb{A}^m)$  holds for all  $m \geq 1$ . Thanks to Proposition 6.2, we have moreover the canonical Morita equivalences

$$(6.1) \quad \mathbf{perf}(X \times \mathbb{A}^m) \simeq \mathbf{perf}(X) \otimes \mathbf{perf}(\mathbb{A}^m) \simeq \mathbf{perf}(X)[t_1, \dots, t_m].$$

As a consequence,  $E_n^b(X) := E_n^b(\mathbf{perf}(X)) \simeq E_n^b(\mathbf{perf}(X)[t_1, \dots, t_m])$  for all  $m \geq 1$ , i.e. the dg category  $\mathbf{perf}(X)$  is  $E_n^b$ -regular. By Theorem 1.2, the dg category  $\mathbf{perf}(X)$  is also  $E_{n-1}^b$ -regular. Hence, using again the above canonical Morita equivalences (6.1) one concludes that the isomorphism  $E_{n-1}^b(X) \simeq E_{n-1}^b(X \times \mathbb{A}^m)$  holds for all  $m \geq 1$ , i.e. that  $X$  is  $E_{n-1}^b$ -regular. This concludes the proof.

**Proposition 6.2.** *Given quasi-compact separated  $k$ -schemes  $X$  and  $Y$ , there is a canonical Morita equivalence*

$$- \boxtimes - : \mathbf{perf}(X) \otimes \mathbf{perf}(Y) \simeq \mathbf{perf}(X \times Y).$$

*Proof.* As proved by Bondal-Orlov in [4, Thm. 3.1.1], the dg categories  $\mathbf{perf}(X)$  and  $\mathbf{perf}(Y)$  admit strong generators  $E_X$  and  $E_Y$ . As a consequence, the (derived) dg  $k$ -algebras of endomorphisms of  $E_X$  and  $E_Y$ , denoted respectively by  $A_X$  and  $A_Y$ , are Morita equivalent to  $\mathbf{perf}(X)$  and  $\mathbf{perf}(Y)$ . Moreover, as proved in [4, Lemma 3.4.1], the perfect complex  $E_X \boxtimes E_Y$  is a strong generator of  $\mathbf{perf}(X \times Y)$  and the induced map  $A_X \otimes A_Y \rightarrow A_{X \times Y}$  is a quasi-isomorphism, where  $A_{X \times Y}$  stands for the (derived) dg  $k$ -algebra of endomorphisms of  $E_X \boxtimes E_Y$ . This achieves the proof.  $\square$

*Remark 6.3.* Given a quasi-compact separated  $k$ -scheme  $X$ , let

$$X[x] := X \times \mathbb{A}^1 \quad X[1/x] := X \times \operatorname{spec}(k[1/x]) \quad X[x, 1/x] := X \times \operatorname{spec}(k[x, 1/x]).$$

Making use of Proposition 6.2, one observes that Theorem 4.2 applied to  $\mathcal{A} = \mathbf{perf}(X)$  reduces to the following exact sequence of abelian groups

$$0 \rightarrow E_n^b(X) \xrightarrow{\Delta} E_n^b(X[x]) \oplus E_n^b(X[1/x]) \xrightarrow{\pm} E_n^b(X[x, 1/x]) \xrightarrow{\partial_3} E_{n-1}^b(X) \rightarrow 0.$$

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